https://www.linkedin.com/feed/update/urn:li:activity:6477426703848341504

4. Prove that for any real numbers $x_1, x_2, \ldots, x_n \in (0, 1/2]$ holds inequality
$\left(\frac{n}{x_1 + x_2 + \ldots + x_n} - 1\right)^n \le \left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\ldots\left(\frac{1}{x_n} - 1\right)$.

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

First we will prove this inequality for $n = 2$, namely inequality

(**1**) $\left(\frac{2}{x+y} - 1\right)^2 \le \left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right)$, $x, y \in (0, 1/2]$.

Since $x + y \le 1$ we obtain $\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right) - \left(\frac{2}{x+y} - 1\right)^2 =$
$\frac{(1-(x+y))(x-y)^2}{xy(x+y)^2} \ge 0$.

Having inequality (**1**) as base of Math Induction and for any $n \ge 3$, assuming that inequality $\left(\frac{k}{x_1 + x_2 + \ldots + x_k} - 1\right)^k \le \left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\ldots\left(\frac{1}{x_k} - 1\right)$ holds for any $2 \le k < n$ we consider two cases:

1. In case $n = 2m$ since by assumption holds inequalities
$\left(\frac{m}{x_1 + x_2 + \ldots + x_k} - 1\right)^m \le \left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\ldots\left(\frac{1}{x_m} - 1\right)$,
$\left(\frac{m}{x_{m+1} + x_{m+2} + \ldots + x_{2m}} - 1\right)^m \le \left(\frac{1}{x_{m+1}} - 1\right)\left(\frac{1}{x_{m+2}} - 1\right)\ldots\left(\frac{1}{x_{2m}} - 1\right)$
and by replacing (x, y) in inequality (**1**) with pair of numbers
$\left(\frac{x_1 + x_2 + \ldots + x_k}{m}, \frac{x_{m+1} + x_{m+2} + \ldots + x_{2m}}{m}\right)$ which belong to $(0, 1/2]$ as well
we obtain inequality $\left(\frac{m}{x_1 + x_2 + \ldots + x_k} - 1\right)\left(\frac{m}{x_{m+1} + x_{m+2} + \ldots + x_{2m}} - 1\right) =$
$\left(\frac{1}{\frac{x_1 + x_2 + \ldots + x_k}{m}} - 1\right)\left(\frac{1}{\frac{x_{m+1} + x_{m+2} + \ldots + x_{2m}}{m}} - 1\right) \ge$
$\left(\frac{2}{\frac{x_1 + x_2 + \ldots + x_k}{m} + \frac{x_{m+1} + x_{m+2} + \ldots + x_{2m}}{m}} - 1\right)^2 = \left(\frac{2m}{x_1 + x_2 + \ldots + x_{2m}} - 1\right)^2$
then $\left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\ldots\left(\frac{1}{x_{2m}} - 1\right) \ge$
$\left(\frac{m}{x_1 + x_2 + \ldots + x_k} - 1\right)^m\left(\frac{m}{x_{m+1} + x_{m+2} + \ldots + x_{2m}} - 1\right)^m \ge \left(\frac{2m}{x_1 + x_2 + \ldots + x_{2m}} - 1\right)^{2m}$;

2. If $n = 2m - 1$, where $m \ge 2$ then applying case 1 to $x_1, x_2, \ldots, x_{2m-1} \in (0, 1/2]$ and
$x_{2m} = \frac{x_1 + x_2 + \ldots + x_{2m-1}}{2m-1} \in (0, 1/2]$ we obtain inequality
$\left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\ldots\left(\frac{1}{x_{2m-1}} - 1\right)\left(\frac{1}{x_{2m}} - 1\right) \ge \left(\frac{2m}{x_1 + x_2 + \ldots + x_{2m-1} + x_{2m}} - 1\right)^{2m} =$
$\left(\frac{2m}{(2m-1)x_{2m} + x_{2m}} - 1\right)^{2m} = \left(\frac{1}{x_{2m}} - 1\right)^{2m}$.

Hence, $\left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\ldots\left(\frac{1}{x_{2m-1}} - 1\right) \ge \left(\frac{1}{x_{2m}} - 1\right)^{2m-1} =$
$\left(\frac{2m-1}{x_1 + x_2 + \ldots + x_{2m-1}} - 1\right)^{2m-1}$.

Thus, by Math Induction proved that inequality
$\left(\frac{n}{x_1 + x_2 + \ldots + x_n} - 1\right)^n \le \left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\ldots\left(\frac{1}{x_n} - 1\right)$ holds for any $n \ge 2$.